

REGION OF VARIABILITY FOR EXPONENTIALLY CONVEX UNIVALENT FUNCTIONS

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ABSTRACT. For $\alpha \in \mathbb{C} \setminus \{0\}$ let $\mathcal{E}(\alpha)$ denote the class of all univalent functions f in the unit disk \mathbb{D} and is given by $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, satisfying

$$\operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} + \alpha z f'(z) \right) > 0 \quad \text{in } \mathbb{D}.$$

For any fixed z_0 in the unit disk \mathbb{D} and $\lambda \in \overline{\mathbb{D}}$, we determine the region of variability $V(z_0, \lambda)$ for $\log f'(z_0) + \alpha f(z_0)$ when f ranges over the class

$$\mathcal{F}_\alpha(\lambda) = \{f \in \mathcal{E}(\alpha) : f''(0) = 2\lambda - \alpha\}.$$

We geometrically illustrate the region of variability $V(z_0, \lambda)$ for several sets of parameters using Mathematica. In the final section of this article we propose some open problems.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane \mathbb{C} . We denote the class of analytic functions in \mathbb{D} by \mathcal{H} which we think of as a topological vector space endowed with the topology of uniform convergence over compact subsets of \mathbb{D} . Let \mathcal{A} denote the family of functions f in \mathcal{H} normalized by $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be in the family \mathcal{S} if it is univalent in \mathbb{D} . Denote by \mathcal{S}^* the subclass of functions $\phi \in \mathcal{A}$ such that ϕ maps \mathbb{D} univalently onto a domain $\Omega = \phi(\mathbb{D})$ that is starlike with respect to the origin. That is, $t\phi(z) \in \phi(\mathbb{D})$ for each $t \in [0, 1]$. It is well known that $\phi \in \mathcal{S}^*$ if and only if

$$\operatorname{Re} \left(\frac{z\phi'(z)}{\phi(z)} \right) > 0, \quad z \in \mathbb{D}.$$

Functions in \mathcal{S}^* are referred to as starlike functions. It is well known that $\phi \in \mathcal{A}$ maps \mathbb{D} univalently onto a convex domain, denoted by $\phi \in \mathcal{C}$, if and only if $z\phi' \in \mathcal{S}^*$. Functions in \mathcal{C} are referred to as normalized convex functions. We refer to the books [2, 3, 4] for a detailed discussion on these two classes.

A function $f \in \mathcal{H}$ is called exponentially convex if f is univalent in \mathbb{D} and $e^{f(z)}$ maps \mathbb{D} onto a convex domain. For $\alpha \in \mathbb{C} \setminus \{0\}$, the family $\mathcal{E}(\alpha)$ of α -exponential functions was introduced in [1]. A function $f \in \mathcal{S}$ is said to be in $\mathcal{E}(\alpha)$ if $F(\mathbb{D})$ is

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a convex domain, where $F(z) = e^{\alpha f(z)}$. Although Arango *et al* [1] introduced and studied exponentially convex functions in 1997, no attempt has been made to study many other properties of the class $\mathcal{E}(\alpha)$ until the present article. In this article, we initiate certain issues related to $\mathcal{E}(\alpha)$. We now recall a number of basic properties of the class $\mathcal{E}(\alpha)$ from [1].

Theorem A. ([1, Theorem 1]) *Let $\alpha \in \mathbb{C} \setminus \{0\}$. A function f is in $\mathcal{E}(\alpha)$ if and only if $\operatorname{Re} P_f(z) > 0$ in \mathbb{D} , where*

$$(1.1) \quad P_f(z) = 1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z).$$

Theorem B. ([1, Theorem 2]) *Let $f \in \mathcal{E}(\alpha)$. Then $f(\mathbb{D})$ is convex in the $\overline{\alpha}$ -direction (and therefore close-to-convex). It is not necessarily starlike univalent.*

Because each $f \in \mathcal{E}(\alpha)$ can be represented in the form

$$e^{\alpha f(z)} = 1 + \alpha g(z), \quad g \in \mathcal{A},$$

as shown in [1], g belongs to the class $\mathcal{C}(\alpha)$ of (normalized) convex univalent functions with $-1/\alpha \notin g(\mathbb{D})$. Here $\mathcal{C}(\alpha)$ denotes the class of convex functions of order α .

Theorem C. ([1, Theorem 3]) *For $\alpha \in \mathbb{C} \setminus \{0\}$ we have*

$$\mathcal{E}(\alpha) = \left\{ \frac{1}{\alpha} \log(1 + \alpha g) : g \in \mathcal{C}(\alpha) \right\}.$$

In [1, Theorem 4], it is also observed that $\mathcal{E}(\alpha) = \emptyset$ if $|\alpha| > 2$ and for $|\alpha| = 2$, $\mathcal{E}(\alpha)$ consists of the functions

$$f(z) = \frac{1}{\alpha} \log \frac{2 + \alpha z}{2 - \alpha z}.$$

Throughout the discussion, we assume that $|\alpha| \leq 2$.

For $f \in \mathcal{F}_\alpha$, we denote by $\log f'$ the single-valued branch of the logarithm of f' with $\log f'(0) = 0$. The Herglotz formula for analytic functions that have positive real part in the unit disk shows that if $f \in \mathcal{E}(\alpha)$, then there exists a unique positive unit measure μ on $(-\pi, \pi]$ such that

$$1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z) = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \quad z \in \mathbb{D}.$$

A computation gives that

$$\log f'(z) + \alpha f(z) = \int_{-\pi}^{\pi} \log \left(\frac{1}{1 - ze^{-it}} \right)^2 d\mu(t),$$

or equivalently we can write

$$\log f'(z) + \alpha f(z) = 2 \int_0^1 \frac{\omega(tz)}{1 - \omega(tz)} \frac{dt}{t}$$

for some $\omega \in \mathcal{B}_0$. Here \mathcal{B}_0 denotes the class of analytic functions ω in \mathbb{D} such that $|\omega(z)| \leq 1$ in \mathbb{D} and $\omega(0) = 0$. Consequently, for each $f \in \mathcal{E}(\alpha)$ there exists an $\omega_f \in \mathcal{B}_0$ of the form

$$(1.2) \quad \omega_f(z) = \frac{P_f(z) - 1}{P_f(z) + 1}, \quad z \in \mathbb{D},$$

and conversely. It is a simple exercise to see that

$$(1.3) \quad P'_f(0) = 2\omega'_f(0) = f''(0) + \alpha.$$

Suppose that $f \in \mathcal{E}(\alpha)$. Then, a simple application of the Schwarz lemma (see for example [2, 5, 6]) shows that

$$|P'_f(0)| = |f''(0) + \alpha| \leq 2,$$

because $|\omega'_f(0)| \leq 1$. That is $f''(0) = 2\lambda - \alpha$ for some $\lambda \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. For $\omega_f \in \mathcal{B}_0$, we define the function $g : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ by

$$(1.4) \quad g(z) = \begin{cases} \frac{\frac{\omega_f(z) - \lambda}{z}}{1 - \overline{\lambda} \frac{\omega_f(z)}{z}} & \text{if } |\lambda| < 1, \\ 0 & \text{if } |\lambda| = 1. \end{cases}$$

Then we see that

$$(1.5) \quad g'(0) = \begin{cases} \frac{\omega''_f(0)}{2(1 - |\lambda|^2)} & \text{if } |\lambda| < 1, \\ 0 & \text{if } |\lambda| = 1. \end{cases}$$

From (1.2) and (1.3), we find that

$$(1.6) \quad 2\omega''_f(0) + (P'_f(0))^2 = P''_f(0).$$

As $f''(0) = 2\lambda - \alpha$, using (1.1) and (1.6) we obtain

$$(1.7) \quad \omega''_f(0) = f'''(0) - 6\lambda(\lambda - \alpha) - 2\alpha^2.$$

From (1.5) and (1.7) we also note that $|g'(0)| \leq 1$ if and only if

$$f'''(0) = 2[(1 - |\lambda|^2)a + 3\lambda(\lambda - \alpha) + \alpha^2]$$

for some $a \in \overline{\mathbb{D}}$. Consequently, for $\lambda \in \overline{\mathbb{D}}$ and for $z_0 \in \mathbb{D}$ fixed, it is natural to introduce

$$\mathcal{F}_\alpha(\lambda) = \{f \in \mathcal{E}(\alpha) : f''(0) = 2\lambda - \alpha\},$$

and

$$V(z_0, \lambda) = \{\log f'(z_0) + \alpha f(z_0) : f \in \mathcal{F}_\alpha(\lambda)\}.$$

From (1.3) and the normalization condition in the class $\mathcal{F}_\alpha(\lambda)$, we observe that $\omega'_f(0) = \lambda$.

In the recent past, several authors (see [10, 13] and references there in) have studied region of variability problems for several subclasses of \mathcal{S} . The main aim of this article is to determine the region of variability $V(z_0, \lambda)$ of $\log f'(z_0) + \alpha f(z_0)$

when f ranges over the class $\mathcal{F}_\alpha(\lambda)$. Our main theorem here is Theorem 2.7 and at the end we graphically illustrate the region of variability for several sets of parameters. Finally, we propose some open problems on exponentially convex univalent functions.

2. THE BASIC PROPERTIES OF $V(z_0, \lambda)$ AND THE MAIN RESULT

To state our main theorem, we need some preparation. For a positive integer p , let

$$(\mathcal{S}^*)^p = \{f = f_0^p : f_0 \in \mathcal{S}^*\}$$

and recall the following result from [13].

Lemma 2.1. *Let f be an analytic function in \mathbb{D} with $f(z) = z^p + \dots$. If*

$$\operatorname{Re} \left(1 + z \frac{f''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D},$$

then $f \in (\mathcal{S}^)^p$.*

Now, we list down some basic properties of $V(z_0, \lambda)$.

Proposition 2.2. *We have*

- (1) $V(z_0, \lambda)$ is a compact subset of \mathbb{C} .
 - (2) $V(z_0, \lambda)$ is a convex subset of \mathbb{C} .
 - (3) For $|\lambda| = 1$ or $z_0 = 0$,
- $$(2.3) \quad V(z_0, \lambda) = \{-2 \log(1 - \lambda z_0)\}.$$
- (4) For $|\lambda| < 1$ and $z_0 \neq 0$, $V(z_0, \lambda)$ has $-2 \log(1 - \lambda z_0)$ as an interior point.

Proof. (1) Since $\mathcal{F}_\alpha(\lambda)$ is a compact subset of \mathcal{H} , it follows that $V(z_0, \lambda)$ is also a compact subset of \mathbb{C} .

(2) If $f_0, f_1 \in \mathcal{F}_\alpha(\lambda)$ and $0 \leq t \leq 1$, then the function f_t satisfying

$$\log f'_t(z) + \alpha f_t(z) = (1-t)(\log f'_0(z) + \alpha f_0(z)) + t(\log f'_1(z) + \alpha f_1(z))$$

is evidently in $\mathcal{F}_\alpha(\lambda)$. Also, because of the representation of f_t , we see easily that the set $V(z_0, \lambda)$ is a convex subset of \mathbb{C} .

(3) If $z_0 = 0$, then (2.3) trivially holds. If $|\lambda| = |\omega'_f(0)| = 1$, then it follows from the Schwarz lemma (see for example [2, 5, 6]) that $\omega_f(z) = \lambda z$, which implies

$$P_f(z) = \frac{1 + \lambda z}{1 - \lambda z}.$$

A simple computation yields

$$\log f'(z) + \alpha f(z) = -2 \log(1 - \lambda z).$$

Consequently,

$$V(z_0, \lambda) = \{-2 \log(1 - \lambda z_0)\}.$$

(4) For $\lambda \in \mathbb{D}$, $z_0 \in \mathbb{D} \setminus \{0\}$, and $a \in \overline{\mathbb{D}}$, we define

$$\delta(z, \lambda) = \frac{z + \lambda}{1 + \overline{\lambda} z}.$$

Recall that $|g'(0)| \leq 1$ if and only if $f'''(0) = 2[(1 - |\lambda|^2)a + 3\lambda(\lambda - \alpha) + \alpha^2]$ for some $a \in \mathbb{D}$. Now by applying the Schwarz lemma to $g(z)$ defined in (1.4) we obtain

$$(2.4) \quad g(z) = az \quad \text{for some } |a| = 1.$$

A simple computation of (2.4) gives

$$(2.5) \quad \log H'_{a,\lambda}(z) + \alpha H_{a,\lambda}(z) = \int_0^z \frac{2\delta(a\zeta, \lambda)}{1 - \delta(a\zeta, \lambda)\zeta} d\zeta, \quad z \in \mathbb{D}.$$

First we claim that $H_{a,\lambda}$ satisfying (2.5) belongs to $\mathcal{F}_\alpha(\lambda)$. For this, we compute

$$1 + \frac{zH''_{a,\lambda}(z)}{H'_{a,\lambda}(z)} + \alpha H'_{a,\lambda}(z) = \frac{1 + \delta(az, \lambda)z}{1 - \delta(az, \lambda)z}.$$

As $\delta(az, \lambda)$ lies in the unit disk \mathbb{D} , $H_{a,\lambda} \in \mathcal{F}_\alpha(\lambda)$ and the claim follows. Also we observe that

$$(2.6) \quad \omega_{H_{a,\lambda}}(z) = z\delta(az, \lambda).$$

Next we claim that the mapping $\mathbb{D} \ni a \mapsto \log H'_{a,\lambda}(z_0) + \alpha H_{a,\lambda}(z_0)$ is a non-constant analytic function of a for each fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{D}$. To do this, we put

$$h(z) = \frac{1}{(1 - |\lambda|^2)} \frac{\partial}{\partial a} \left\{ \log H'_{a,\lambda}(z) + \alpha H_{a,\lambda}(z) \right\} \Big|_{a=0}.$$

A computation gives

$$h(z) = 2 \int_0^z \frac{\zeta}{(1 - \lambda\zeta)^2} d\zeta = z^2 + \dots$$

from which it is easy to see that

$$\operatorname{Re} \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} = \operatorname{Re} \left\{ \frac{2}{1 - \lambda z} \right\} > 1, \quad z \in \mathbb{D}.$$

By Lemma 2.1 there exists a function $h_0 \in \mathcal{S}^*$ with $h = h_0^2$. The univalence of h_0 together with the condition $h_0(0) = 0$ implies that $h(z_0) \neq 0$ for $z_0 \in \mathbb{D} \setminus \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto \log H'_{a,\lambda}(z_0) + \alpha H_{a,\lambda}(z_0)$ is a non-constant analytic function of a and hence, it is an open mapping. Thus, $V(z_0, \lambda)$ contains the open set $\{\log H'_{a,\lambda}(z_0) + \alpha H_{a,\lambda}(z_0) : |a| < 1\}$. In particular,

$$\log H'_{0,\lambda}(z_0) + \alpha H_{0,\lambda}(z_0) = -2 \log(1 - \lambda z_0)$$

is an interior point of $\{\log H'_{a,\lambda}(z_0) + \alpha H_{a,\lambda}(z_0) : a \in \mathbb{D}\} \subset V(z_0, \lambda)$. \square

We remark that, since $V(z_0, \lambda)$ is a compact convex subset of \mathbb{C} and has nonempty interior, the boundary $\partial V(z_0, \lambda)$ is a Jordan curve and $V(z_0, \lambda)$ is the union of $\partial V(z_0, \lambda)$ and its inner domain. Now we state our main result and the proof will be presented in Section 3.

Theorem 2.7. For $\lambda \in \mathbb{D}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V(z_0, \lambda)$ is the Jordan curve given by

$$(-\pi, \pi] \ni \theta \mapsto \log H'_{e^{i\theta}, \lambda}(z_0) + \alpha H_{e^{i\theta}, \lambda}(z_0) = \int_0^{z_0} \frac{2\delta(e^{i\theta}\zeta, \lambda)}{1 - \delta(e^{i\theta}\zeta, \lambda)\zeta} d\zeta.$$

If $f(z_0) = H_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{F}_\alpha(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f(z) = H_{e^{i\theta}, \lambda}(z)$.

3. PREPARATION FOR THE PROOF OF THEOREM 2.7

Proposition 3.1. For $f \in \mathcal{F}_\alpha(\lambda)$ and $\lambda \in \mathbb{D}$ we have

$$(3.2) \quad \left| \frac{f''(z)}{f'(z)} + \alpha f'(z) - c(z, \lambda) \right| \leq r(z, \lambda), \quad z \in \mathbb{D},$$

where

$$c(z, \lambda) = \frac{2[\lambda(1 - |z|^2) + \bar{z}(|z|^2 - |\lambda|^2)]}{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))}, \quad \text{and}$$

$$r(z, \lambda) = \frac{2(1 - |\lambda|^2)|z|}{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))}.$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta}, \lambda}$ for some $\theta \in \mathbb{R}$.

Proof. Set $\gamma = 0$, $\beta = 0$ in [8, Proposition 4.1]. Then [8, Proposition 4.1] takes the following form

$$(3.3) \quad \left| 1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z) - \frac{(1 + \lambda z)(1 - \bar{\lambda}\bar{z}) + |z|^2(\bar{z} - \lambda)(\bar{\lambda} + z)}{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))} \right|$$

$$\leq \frac{2(1 - |\lambda|^2)|z|^2}{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))}, \quad z \in \mathbb{D}.$$

A simplification of (3.3) gives (3.2). □

The choice $\lambda = 0$ gives the following result which may need a special mention.

Corollary 3.4. For $f \in \mathcal{F}_\alpha(0)$ we have

$$(3.5) \quad \left| \frac{f''(z)}{f'(z)} + \alpha f'(z) - \frac{2|z|^2\bar{z}}{1 - |z|^4} \right| \leq \frac{2|z|}{1 - |z|^4}, \quad z \in \mathbb{D}.$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta}, 0}$ for some $\theta \in \mathbb{R}$.

If $f \in \mathcal{F}_\alpha(0)$, then by (3.5) we obtain

$$(1 - |z|^2) \left| \frac{f''(z)}{f'(z)} + \alpha f'(z) \right| \leq 2|z|.$$

Corollary 3.6. Let $\gamma: z(t)$, $0 \leq t \leq 1$ be a C^1 -curve in \mathbb{D} with $z(0) = 0$ and $z(1) = z_0$. Then we have

$$V(z_0, \lambda) \subset \{w \in \mathbb{C}: |w - C(\lambda, \gamma)| \leq R(\lambda, \gamma)\},$$

where

$$C(\lambda, \gamma) = \int_0^1 c(z(t), \lambda) z'(t) dt \quad \text{and} \quad R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda) |z'(t)| dt.$$

Proof. Since the proof of Corollary 3.6 follows from [9], we omit the details. \square

For the proof of our next result, we need the following lemma.

Lemma 3.7. [9] *For $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{D}$ the function*

$$G(z) = \int_0^z \frac{e^{i\theta} \zeta}{\{1 + (\bar{\lambda} e^{i\theta} - \lambda) \zeta - e^{i\theta} \zeta^2\}^2} d\zeta, \quad z \in \mathbb{D},$$

has a double zero at the origin and no zeros elsewhere in \mathbb{D} . Furthermore there exists a starlike univalent function G_0 in \mathbb{D} such that $G = e^{i\theta} G_0^2$ and $G_0(0) = G'_0(0) - 1 = 0$.

Proposition 3.8. *Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have $\log H'_{e^{i\theta}, \lambda}(z_0) + \alpha H_{e^{i\theta}, \lambda}(z_0) \in \partial V(z_0, \lambda)$. Furthermore, if $\log f'(z_0) + \alpha f(z_0) = \log H'_{e^{i\theta}, \lambda}(z_0) + \alpha H_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{F}_\alpha(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f = H_{e^{i\theta}, \lambda}$.*

Proof. For a proof we refer to [8, Proposition 4.11] with

$$P(z) = 1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z). \quad \square$$

Proof of Theorem 2.7 Although the proof of Theorem 2.7 is similar to that of the main theorem in [8, Theorem 5.1], for the sake of completeness we include the proof. We need to prove that the closed curve

$$(3.9) \quad (-\pi, \pi] \ni \theta \mapsto \log H'_{e^{i\theta}, \lambda}(z_0) + \alpha H_{e^{i\theta}, \lambda}(z_0)$$

is simple. Suppose that

$$\log H'_{e^{i\theta_1}, \lambda}(z_0) + \alpha H_{e^{i\theta_1}, \lambda}(z_0) = \log H'_{e^{i\theta_2}, \lambda}(z_0) + \alpha H_{e^{i\theta_2}, \lambda}(z_0)$$

for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$. Then, from Proposition 3.8, we have

$$H_{e^{i\theta_1}, \lambda} = H_{e^{i\theta_2}, \lambda}.$$

From (2.6) we have

$$\tau\left(\frac{\omega_{H_{e^{i\theta}, \lambda}}}{z}, \lambda\right) = \frac{(1 - \bar{\lambda})e^{i\theta}z + \lambda - \bar{\lambda}}{1 - \lambda^2 - (\lambda - \bar{\lambda})e^{i\theta}z}, \quad \tau(z, \lambda) = \frac{z - \bar{\lambda}}{1 - \lambda z}.$$

That is

$$\frac{(1 - \bar{\lambda})e^{i\theta_1}z + \lambda - \bar{\lambda}}{1 - \lambda^2 - (\lambda - \bar{\lambda})e^{i\theta_1}z} = \frac{(1 - \bar{\lambda})e^{i\theta_2}z + \lambda - \bar{\lambda}}{1 - \lambda^2 - (\lambda - \bar{\lambda})e^{i\theta_2}z}$$

and a simple computation yields

$$e^{i\theta_1}z = e^{i\theta_2}z$$

which is a contradiction for the choice of θ_1 and θ_2 . Thus, the curve must be simple. Since $V(z_0, \lambda)$ is a compact convex subset of \mathbb{C} and has nonempty interior, the boundary $\partial V(z_0, \lambda)$ is a simple closed curve. From Proposition 3.8, the curve

$\partial V(z_0, \lambda)$ contains the curve (3.9). Recall the fact that a simple closed curve cannot contain any simple closed curve other than itself. Thus, the curve $\partial V(z_0, \lambda)$ is given by (3.9). \square

4. GEOMETRIC VIEW OF THEOREM 2.7

Using Mathematica 7, we describe the boundary of the set $V(z_0, \lambda)$. Here we give the Mathematica program which is used to plot the boundary of the set $V(z_0, \lambda)$. We refer to [12] for the basic concepts on Mathematica programming. The short notations in this program are of the form: “z0 for z_0 ”, “lam for λ ”.

```
Remove["Global`*"];

z0 = Random[] Exp[I Random[Real, {-Pi, Pi}]]
lam = Random[] Exp[I Random[Real, {-Pi, Pi}]]

Q[lam_, the_] := (2 (Exp[I*the]*z + lam))/
(1 + (Conjugate[lam]*Exp[I*the] - lam)*z - Exp[I*the]*z^2)

myf[lam_, the_, z0_] :=NIntegrate[Q[lam, the], {z, 0, z0},
PrecisionGoal -> 2]

image = ParametricPlot[With[{q = myf[lam, the, z0]},
{Re[q], Im[q]}], {the, -Pi, Pi}]
```

Table 1

Figure	z_0	λ
1	0.0230875+0.00517512i	0.175557-0.225417i
2	0.147076+0.0913164i	0.0748874+0.0476965i
3	-0.819143-0.551002i	0.722765+0.433556i
4	0.757794-0.598957i	-0.308071-0.32103i
5	-0.414782-0.377338i	0.196381-0.500501i
6	0.386456-0.316514i	-0.236285+0.235873i
7	0.419565+0.478471i	0.242605+0.097106i
8	0.754872+0.0830025i	0.130907+0.931628i

The Figures from 1 to 8 give the geometric view of the region of variability $V(z_0, \lambda)$ for some sets of parameters $z_0 \in \mathbb{D}$ and $\lambda \in \mathbb{D}$. We observe from Figures 1 and 2 that the regions of variability $V(z_0, \lambda)$ are very small for some sets of parameters z_0 and λ whereas Figures 7 and 8 show that the regions of variability $V(z_0, \lambda)$ are relatively large for some particular sets of parameters. With the help of Mathematica 7, we have drawn the curves $\partial V(z_0, \lambda)$ for various values of z_0 and λ and observed that the regions of variability for exponentially convex functions are of the small size.

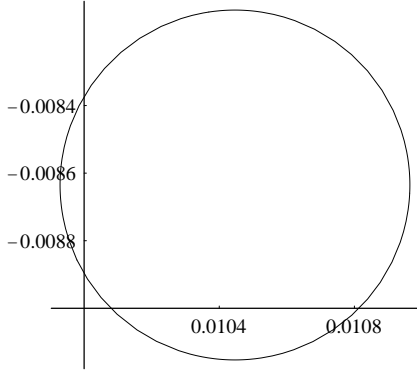


FIGURE 1.

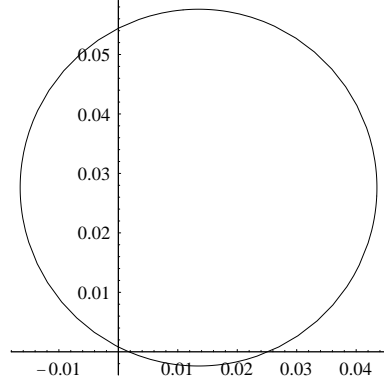


FIGURE 2.

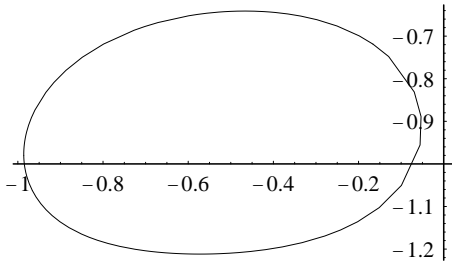


FIGURE 3.

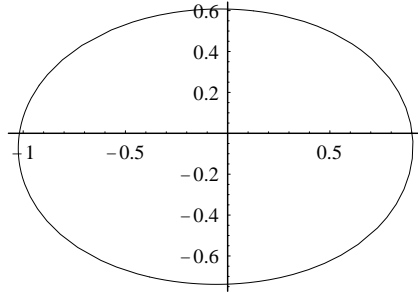


FIGURE 4.

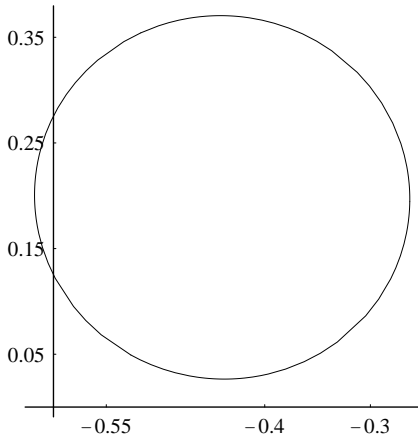


FIGURE 5.

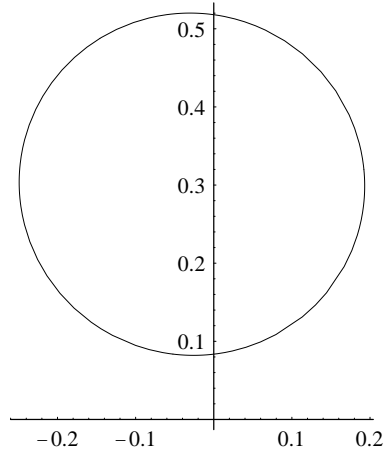


FIGURE 6.

The above pictures are evident to Proposition 2.2 that the regions bounded by the curves $\partial V(z_0, \lambda)$ are compact and convex subsets of \mathbb{C} .

5. OPEN PROBLEMS

- (1) For $f \in \mathcal{E}(\alpha)$, what are the sharp lower and upper bounds of $|f(z)|$ and $|f'(z)|$ for $z \in \mathbb{D}$?

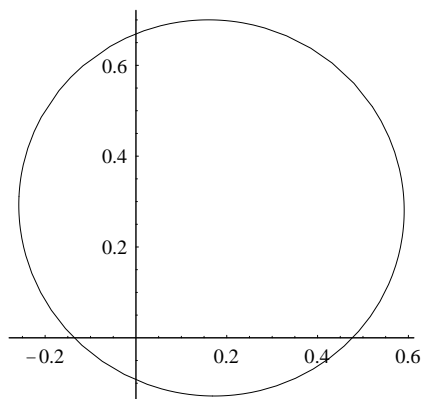


FIGURE 7.

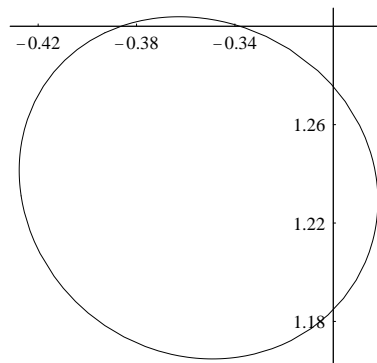


FIGURE 8.

- (2) Let $f \in \mathcal{E}(\alpha)$ and be given by $f(z) = z + \sum_{n=0}^{\infty} a_n z^n$. Then what are the sharp coefficient bounds for $|a_n|$ for $n \geq 2$?

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